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Finite Element Solution of Optimal Control Problems with State-Control Inequality Constraints

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Introduction

A NEW method is presented for solving optimal control problems with inequality constraints that are functions of both control and state variables. The method is based on the weak Hamiltonian formulation derived in Ref. 1 and uses finite elements in time. The extension of the weak formulation is already outlined in Ref. 2 where the methodology is applied to a two-stage rocket trajectory optimization problem. This Note is intended to fill in some of the details omitted in Ref. 2 concerning the solution of problems with state-control inequality constraints. Herein it is shown that the formulation does not require element quadrature, and it produces a sparse system of nonlinear algebraic equations. Since these algebraic equations may be derived before specifying the problem to be solved, the formulation is conducive to development of a general purpose computational environment for the solution of a large class of optimal control problems. After the derivation is given, a simple example problem is presented. The numerical results are compared with the exact solution. Of particular interest is the performance in terms of execution time and accuracy vs the number of elements used to represent the time span of the problem.

General Development

It is desired to develop a solution strategy for optimal control problems with inequality constraints based on finite elements in time.[1] In an attempt to make the solution scheme as general as possible, all strong boundary conditions (i.e., boundary conditions that require the variational quantities to be zero) will be transformed into natural boundary conditions (i.e., boundary conditions that are determined by setting the coefficient of a variational quantity to zero). This is done so that the shape functions can be chosen from a less restrictive class of functions, which enables one to choose the same shape functions for every optimal control problem.

The idea of transforming the strong boundary conditions to natural boundary conditions[3] revolves around adjoining a constraint equation to the performance index with an unknown Lagrange multiplier. The variation of the performance index is then taken in a straightforward manner. Through appropriate integration by parts, it is possible to show that the Euler-Lagrange equations are identical to those derived in classical textbooks[4] and that the boundary conditions are the same, only stated weakly instead of strongly.

Consider a system defined by a set of n states x and a set of m controls u. Let the system be governed by a set of state equations of the form $\dot{x} = f(x,u,t)$. In this Note, the class of problems to be considered is limited to the case where x is continuous, but there may be discontinuities in the control u. These discontinuities may be the result of inequality constraints present in the problem. Elements of the performance index J_0 may be denoted by an integrand $L(x,u,t)$ and discrete functions of the states and time $\phi[x(t),t]$ at the initial and final times t_0 and t_f. In addition, any constraints imposed on the states and time at the initial and final times may be placed in sets of functions $\psi[x(t),t]$. These constraints may be adjoined to the performance index by discrete Lagrange multipliers ν defined at t_0 and t_f. Similarly, the state equations may be adjoined to the performance index with a set of Lagrange multiplier functions $\lambda(t)$ that will be referred to as costates. Finally, suppose that g is a $p \times 1$ column matrix of constraints on the controls and states of the form

$$g(x,u,t) \leq 0 \tag{1}$$

One way of handling inequality constraints is to use a "slack" variable.[5] The idea is that if $g \leq 0$, then g plus some positive number (i.e., the slack variable) is equal to zero. Thus denoting the slack variable by k^2, then the following $p \times 1$ column matrices for K and δK, the variation of K, may be defined:

$$K = \lfloor k_1^2 \;\; k_2^2 \;\; \cdots \;\; k_p^2 \rfloor^T$$

$$\delta K = \lfloor 2k_1\delta k_1 \;\; 2k_2\delta k_2 \;\; \cdots \;\; 2k_p\delta k_p \rfloor^T \tag{2}$$

Now, from Eq. (1)

$$g(x,u,t) + K = 0 \tag{3}$$

Equation (3) will also be adjoined to the performance index J_0 by using p Lagrange multiplier functions $\mu(t)$. The performance index now takes the form

$$J_0 = \int_{t_0}^{t_f} [L(x,u,t) + \lambda^T(f - \dot{x}) + \mu^T(g + K)] \, dt + \Phi|_{t_0}^{t_f} \tag{4}$$

where $\Phi = \phi[x(t),t] + \nu^T\psi[x(t),t]$. The constraints to be adjoined to the previous J_0 to transform the strong boundary conditions to weak boundary conditions[3] are that the states be continuous near the initial and final times. Introducing

$$x|_{t_0} \triangleq \lim_{t \to t_0^+} x(t) \quad \text{and} \quad x|_{t_f} \triangleq \lim_{t \to t_f^-} x(t) \tag{5}$$

and

$$\hat{x}_0 = \hat{x}|_{t_0} \triangleq x(t_0) \quad \text{and} \quad \hat{x}_f = \hat{x}|_{t_f} \triangleq x(t_f) \tag{6}$$

one can weakly enforce continuity by adjoining $\alpha^T(x - \hat{x})|_{t_0}^{t_f}$ to J_0 where α is a set of discrete unknown Lagrange multipliers

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defined only at t_0 and t_f. The new performance index is

$$J = \int_{t_0}^{t_f} [L(x,u,t) + \lambda^T(f - \dot{x}) + \mu^T(g + K)]\, dt + \Phi|_{t_0}^{t_f} + \alpha^T(x - \hat{x})|_{t_0}^{t_f} \tag{7}$$

To derive the weak principle, it is necessary to take the first variation of J and set it equal to zero. For simplicity, the next derivation is for the case of fixed-final time. The case for free-final time is discussed after this derivation. For notational convenience, the following variables are introduced:

$$\hat{\lambda}_0 = \hat{\lambda}|_{t_0} = \frac{\partial \Phi}{\partial x}\bigg|_{t_0} \quad \text{and} \quad \hat{\lambda}_f = \hat{\lambda}|_{t_f} = \frac{\partial \Phi}{\partial x}\bigg|_{t_f} \tag{8}$$

Also, as is shown in Ref. 1, the Lagrange multiplier α can be chosen so that $\delta\alpha = \delta\lambda$.

The first variation of J is

$$\delta J = \int_{t_0}^{t_f} \left\{ \delta\lambda^T(f - \dot{x}) - \delta\dot{x}^T\lambda + \delta x^T \left[\left(\frac{\partial L}{\partial x}\right)^T + \left(\frac{\partial f}{\partial x}\right)^T \lambda + \left(\frac{\partial g}{\partial x}\right)^T \mu \right] + \delta u^T \left[\left(\frac{\partial L}{\partial u}\right)^T + \left(\frac{\partial f}{\partial u}\right)^T \lambda + \left(\frac{\partial g}{\partial u}\right)^T \mu \right] + \delta\mu^T(g + K) + \delta K^T\mu \right\} dt + \delta\nu^T\psi|_{t_0}^{t_f} + \delta x^T\hat{\lambda}|_{t_0}^{t_f} + \delta\lambda^T(x - \hat{x})|_{t_0}^{t_f} + \lambda^T(\delta x - \delta\hat{x})|_{t_0}^{t_f} = 0 \tag{9}$$

The admissible variations of the states are chosen to be continuous at the initial and final times and therefore $(\delta x - \delta\hat{x})|_{t_0}^{t_f} = 0$. Furthermore, it is noted that, for most problems, the initial conditions are given for all n states and thus, in accordance with Eq. (8), all of the initial costates are unknown. Therefore, instead of treating elements of ν at $t = t_0$ as unknowns and replacing $\hat{\lambda}|_{t_0}$ with these unknowns, $\hat{\lambda}|_{t_0}$ will be treated as unknowns and the $\delta\nu|_{t_0}$ equations will be eliminated from the weak principle. Finally, the weak principle is obtained by integrating the $\dot{x}$ term in Eq. (9) by parts. Denoting the variations of the variables at the initial and final times with subscripts 0 and f, then the resulting equation is

$$\int_{t_0}^{t_f} \left\{ -\delta\dot{x}^T\lambda + \delta\lambda^T f + \delta\dot{\lambda}^T x + \delta x^T \left[\left(\frac{\partial L}{\partial x}\right)^T + \left(\frac{\partial f}{\partial x}\right)^T \lambda + \left(\frac{\partial g}{\partial x}\right)^T \mu \right] + \delta u^T \left[\left(\frac{\partial L}{\partial u}\right)^T + \left(\frac{\partial f}{\partial u}\right)^T \lambda + \left(\frac{\partial g}{\partial u}\right)^T \mu \right] + \delta\mu^T(g + K) + \delta K^T\mu \right\} dt + \delta\nu^T\psi|_{t_f} + \delta x_f^T\hat{\lambda}_f - \delta x_0^T\hat{\lambda}_0 - \delta\lambda_f^T\hat{x}_f + \delta\lambda_0^T\hat{x}_0 = 0 \tag{10}$$

This is the governing equation for the weak Hamiltonian method for fixed-time problems with inequality constraints that are functions of both state and control variables. It is easily shown by integrating the $\delta\dot{x}$ and $\delta\dot{\lambda}$ terms by parts in Eq. (10) that all of the Euler-Lagrange equations are the same as in Ref. 4 and that all boundary conditions are now of the natural type. As is shown in Ref. 1, when the final time is allowed to vary, Eq. (10) remains unchanged except that one term is added given by

$$\delta t_f \left(L + \lambda^T f + \frac{\partial \phi}{\partial t} + \nu^T \frac{\partial \psi}{\partial t} \right) \bigg|_{t_f} \tag{11}$$

Note that in this term values for u are required at t_f. To obtain u, one must also find the values of K and μ at t_f. These unknowns are found by setting the coefficients of δu_f^T, δK_f^T, and $\delta\mu_f^T$ equal to zero in the following:

$$\delta u_f^T \left[\left(\frac{\partial L}{\partial u}\right)^T + \left(\frac{\partial f}{\partial u}\right)^T \lambda + \left(\frac{\partial g}{\partial u}\right)^T \mu \right] \bigg|_{t_f}$$

$$\delta\mu_f^T(g + K)\bigg|_{t_f}$$

$$\delta K_f^T \mu \bigg|_{t_f} \tag{12}$$

Thus, the formulation has now been developed to handle fixed- and free-time problems with inequality constraints that are functions of both state and control variables.

Finite Element Discretization

Let the time interval from t_0 to t_f be broken into N elements. The nodal values of time for these elements are t_i for $i = 1, \ldots, N + 1$ where $t_0 = t_1$ and $t_f = t_{N+1}$. A nondimensional elemental time τ is defined as

$$\tau = \frac{t - t_i}{t_{i+1} - t_i} = \frac{t - t_i}{\Delta t_i} \tag{13}$$

Since one derivative of δx and $\delta\lambda$ appears in Eq. (10), linear shape functions for δx and $\delta\lambda$ may be chosen. Since no derivatives of x or λ appear, piecewise constant shape functions for x and λ are chosen. These shape functions are taken to be

$$\delta x = \delta x_i(1 - \tau) + \delta x_{i+1}\tau \tag{14}$$

and

$$x = \hat{x}_i \quad \text{if} \quad \tau = 0$$

$$x = \bar{x}_i \quad \text{if} \quad 0 < \tau < 1$$

$$x = \hat{x}_{i+1} \quad \text{if} \quad \tau = 1 \tag{15}$$

and similarly for $\delta\lambda$ and λ. It is important to understand that $\hat{x}_1 = \hat{x}_0 = x(t_0)$, $\hat{\lambda}_1 = \hat{\lambda}_0 = \lambda(t_0)$, $\hat{x}_{N+1} = \hat{x}_f = x(t_f)$, and $\hat{\lambda}_{N+1} = \hat{\lambda}_f = \lambda(t_f)$.

In addition, since the time derivatives of u, K, μ, δu, δK, and $\delta\mu$ do not appear in the formulation, then

$$u = \bar{u}_i, \quad K = \bar{K}_i, \quad \mu = \bar{\mu}_i$$

$$\delta u = \delta\bar{u}_i, \quad \delta K = \delta\bar{K}_i, \quad \delta\mu = \delta\bar{\mu}_i \tag{16}$$

By substituting Eq. (13) and the shape functions described above into Eq. (10), and carrying out the element quadrature over τ from 0 to 1, a general algebraic form of the Hamiltonian weak principle is obtained. Note that if the time t does not appear explicitly in the problem formulation, then all integration is *exact* and can be done by inspection. If t does appear explicitly, then t may be approximated by a constant value over each element and the integration may still be done by inspection. The latter case occurs in the example problem presented shortly. For N elements, there are $2n(N + 1) + mN + q + 2Np$ equations and $2n(N + 2) + mN + q + 2Np$ unknowns. Therefore, $2n$ of the $4n$ endpoint values for the states and costates ($\hat{x}_0$, $\hat{\lambda}_0$, $\hat{x}_f$, and $\hat{\lambda}_f$) must be specified. In general, $\hat{x}_0$ (the initial conditions) is known in accordance with physical constraints. Also, $\hat{\lambda}_f$ can be specified in terms of other unknowns with the use of Eq. (8). Now there are the same number of equations as unknowns. These equations may be used for any optimal control problem of the form specified.

Normally, the resulting nonlinear algebraic equations can be solved by expressing the Jacobian explicitly and using a Newton-Raphson solution procedure. For the example problem that follows, the iteration procedure will converge quickly for a small number of elements with a trivial initial guess. Then the answers obtained for a small number of elements can be used to generate initial guesses for a higher number of elements.

Although the nodal values $\hat{x}_i$ and $\hat{\lambda}_i$ for $2<i<N$ (on the interior of the time interval) do not appear in the algebraic equations, their values can be easily recovered after the solution, since the element values ($\bar{x}_i$ and $\bar{\lambda}_i$) are just the mean of the nodal values. Although the shape functions for u, K, and μ only define a constant value within the element, values for these variables at additional points are also available. For instance, once the nodal values for the states and costates are found, then one may use the optimality condition ($\partial H/\partial u = 0$), the constraint equations ($g + K = 0$), and the condition that either k or μ be zero ($k\mu = 0$) to solve for u, K, and μ at the nodal point. This procedure is used in the following example problem.

Example

This example is taken from Sec. 3.8 of Ref. 4. The problem is to minimize

$$J = \frac{1}{2}x(T)^2 + \frac{1}{2}\int_0^T u^2 \, dt \tag{17}$$

where $T = 10$, x and u are scalars, and the initial condition is $\hat{x}_0 = -5355692/268515$. The state equation is

$$\dot{x} = h(t)u \quad \text{with} \quad h(t) = 1 + t - \tfrac{3}{17}t^2 \tag{18}$$

The following two control inequality constraints are imposed:

$$g_1 = u - 1 \le 0$$
$$g_2 = -(u + 1) \le 0 \tag{19}$$

The exact solution is found to be $x(T) = -17/39$ and for the control

$$\begin{aligned} u(t) &= -x(T)h(t) && \text{for} \quad 0 \le t \le 2 \\ u(t) &= 1 && \text{for} \quad 2 \le t \le 11/3 \\ u(t) &= -x(T)h(t) && \text{for} \quad 11/3 \le t \le 8 \\ u(t) &= -1 && \text{for} \quad 8 \le t \le 10 \end{aligned} \tag{20}$$

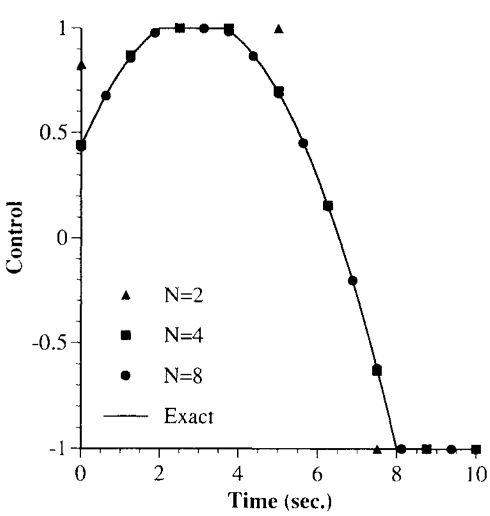

Fig. 1 Control vs time for example problem.

The algebraic equations that come from the weak principle can be verified to be

$$\hat{\lambda}_0 - \bar{\lambda}^{(1)} = 0$$
$$\bar{\lambda}^{(i)} - \bar{\lambda}^{(i+1)} = 0 \quad \text{for} \quad i = 1, 2, \ldots, N-1$$
$$\bar{\lambda}^{(N)} - \hat{x}_f = 0 \tag{21}$$

for the δx coefficients,

$$\left.\begin{aligned} \bar{u}^{(i)} + \bar{\lambda}^{(i)} h[\bar{t}^{(i)}] + \bar{\mu}_1^{(i)} - \bar{\mu}_2^{(i)} &= 0 \\ \bar{k}_1^{(i)}\bar{\mu}_1^{(i)} &= 0 \\ \bar{k}_2^{(i)}\bar{\mu}_2^{(i)} &= 0 \\ \bar{u}^{(i)} - 1 + \bar{k}_1^{(i)2} &= 0 \\ -\bar{u}^{(i)} - 1 + \bar{k}_2^{(i)2} &= 0 \end{aligned}\right\} \quad \text{for } i = 1, 2, \ldots, N \tag{22}$$

for the δu, δK, and $\delta\mu$ coefficients, and

$$\bar{x}^{(1)} - \frac{\Delta t}{2} h[\bar{t}^{(1)}]\bar{u}^{(1)} = \hat{x}_0$$

$$\bar{x}^{(i+1)} - \bar{x}^{(i)} - \frac{\Delta t}{2}\Big\{h[\bar{t}^{(i)}]\bar{u}^{(i)} + h[\bar{t}^{(i+1)}]\bar{u}^{(i+1)}\Big\} = 0 \quad \text{for} \quad i = 1, 2, \ldots, N-1$$

$$-\bar{x}^{(N)} - \frac{\Delta t}{2} h[\bar{t}^{(N)}]\bar{u}^{(N)} + \hat{x}_f = 0 \tag{23}$$

for the $\delta\lambda$ coefficients. Note that $\bar{t}^{(i)}$ is an average time value for the ith element and (if $\Delta t_i = \Delta t = t_f/N$ for all i) can be expressed as

$$\bar{t}^{(i)} = \frac{2i - 1}{2}\Delta t \quad \text{for} \quad i = 1, 2, \ldots, N \tag{24}$$

Recall from Ref. 4 that one of the additional necessary conditions for problems with control constraints is that the multipliers be greater than or equal to zero for a minimizing problem. Therefore, in practice, the multipliers μ appearing in the first of Eqs. (22) are squared to insure their positivity. Further recall that if the constraint is not violated, then $\mu = 0$. This condition is satisfied by the second and third equation in Eq. (22), which implies that either k or μ is zero for each element.

It is readily apparent from the equations shown in Eq. (21) that all of the costate variables are equal to $\hat{x}_f$. Therefore, these equations were eliminated and all costates that occurred in the remaining equations were replaced with $\hat{x}_f$. The remaining $6N + 1$ algebraic equations were solved using a Newton-Raphson method and a FORTRAN code written on a SUN 3/260. The sparse, linearized equations are solved using subroutine MA28 from the Harwell subroutine library.[6] This subroutine takes advantage of sparsity that leads to great computational savings.

Table 1 shows the convergence rate of $\hat{x}_f = x(T)$, the elapsed computer time for the first five iterations, and the

Table 1 $x(T)$, elapsed computer time, and percent sparsity of Jacobian vs the number of elements N

N	$x(T)$	Time, s	Sparsity, %
1	−4.0632	0.42	65.3
2	−0.82795	0.44	80.5
4	−0.44065	0.66	89.6
8	−0.43360	0.76	94.6
16	−0.43928	1.03	97.3
32	−0.43588	1.52	98.6
Exact	−0.43590	—	—

percentage of zeroes in the Jacobian (i.e., the sparsity) vs the number of elements. The $x(T)$ column shows that the 32-element case has almost converged on the exact solution. (Recall that the exact solution is $-17/39 = -0.43590$.) Note further that the approximate $x(T)$ is not an upper bound of the exact value, which is common in mixed formulations. The third column of Table 1 gives the elapsed computer time for five iterations. It is easily seen that there is a modest increase in computer time with an increase in the number of elements. Note that in some cases a converged answer is found in five or fewer iterations. This is because the answers obtained from a small number of elements (say two or four) may be interpolated to generate initial guesses for a higher number of elements. Thus, it is possible to solve a 16- or 32-element case in about 1.5 s. Finally, the extremely sparse structure of the Jacobian is demonstrated in the last column. This strongly encourages the use of a smart sparse matrix solver such as MA28. This subroutine leads to quicker solutions and tremendous savings in memory allocation since only the nonzeroes of the Jacobian need be stored.

Results for the control u are shown in Fig. 1 for two, four, and eight elements and the exact solution. Note that although the two-element case does not define the constraint boundaries very accurately, it is accurate enough to generate guesses for the four-element case. Thus, in a problem with many constrained and unconstrained arcs, a small number of elements could still be used to generate guesses for a higher number of elements. Also, it is interesting to note that as few as four elements have essentially converged on the exact solution.

Conclusions

In this Note, it has been shown that the weak Hamiltonian finite element formulation is amenable to the solution of optimal control problems with inequality constraints that are functions of both state and control variables. Difficult problems may be treated due to the ease with which algebraic equations can be generated before having to specify the problem. These algebraic equations yield very accurate solutions. Furthermore, due to the sparse structure of the resulting Jacobian, computer solutions may be obtained quickly when the sparsity is exploited.

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Equivalence of Two Classes of Dual-Spin Spacecraft Spinup Problems

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Introduction

SPINUP problems for dual-spin spacecraft have been investigated by numerous authors.[1-6] Usually researchers focus on a particular model; for example, Gebman and Mingori[1] and Guelman[4] specifically restricted their attention to the attitude recovery problem for prolate spacecraft. Others have dealt with oblate, prolate, and intermediate spacecraft separately, for example, Hubert.[3] In this Note we give a symmetry transformation for axial dual spinners that allows prolate spacecraft to be treated as oblate. This transformation is quite simple and should prove useful in future investigations of spinup problems.

We begin, of course, with the equations of motion and definitions for the terms oblate, prolate, and intermediate. Following a brief discussion of the transformation, we give an example relating two spacecraft. We then discuss the significance of this result and suggest some possible applications.

Equations of Motion

In the first approximation, a dual-spin spacecraft is usually modeled as two rigid bodies: a platform $\mathcal{P}$ and a rotor $\mathcal{R}$, connected by a shaft that allows relative rotation between them. We denote the system as $\mathcal{P}+\mathcal{R}$ (see Fig. 1). In this work, $\mathcal{P}$ is asymmetric, whereas $\mathcal{R}$ is axisymmetric about the axis of relative rotation e_1, which is a principal axis of $\mathcal{P}+\mathcal{R}$. An internal motor is used to provide an equal and opposite torque to each body along the connecting shaft.

The differential equations for the system $\mathcal{P}+\mathcal{R}$ with no external torque are[7]

$$\dot{h}_1 = \frac{I_2 - I_3}{I_2 I_3} h_2 h_3 \tag{1}$$

$$\dot{h}_2 = \left(\frac{I_3 - I_p}{I_3 I_p} h_1 - \frac{h_a}{I_p}\right) h_3 \tag{2}$$

$$\dot{h}_3 = \left(\frac{I_p - I_2}{I_2 I_p} h_1 + \frac{h_a}{I_p}\right) h_2 \tag{3}$$

$$\dot{h}_a = g_a \tag{4}$$

where

$h_a = I_s(\omega_s + \omega_1)$ = angular momentum of $\mathcal{R}$ about e_1
$h_1 = I_1\omega_1 + I_s\omega_s$ = angular momentum of $\mathcal{P}+\mathcal{R}$ about e_1
$h_i = I_i\omega_i$ = angular momentum of $\mathcal{P}+\mathcal{R}$ about e_i $(i=1,2,3)$
I_i = moment of inertia of $\mathcal{P}+\mathcal{R}$ about e_i $(i=1,2,3)$
I_s = moment of inertia of $\mathcal{R}$ about e_1
$I_p = I_1 - I_s$ = moment of inertia of $\mathcal{P}$ about e_1
ω_i = angular velocity of $\mathcal{P}$ about e_i $(i=1,2,3)$
ω_s = angular velocity of $\mathcal{R}$ about e_1 relative to $\mathcal{P}$
g_a = torque applied by $\mathcal{P}$ on $\mathcal{R}$ about e_1
e_i = principal axes of $\mathcal{P}+\mathcal{R}$ $(i=1,2,3)$

It is evident from Eqs. (1-3) that the magnitude of I_p relative to I_2 and I_3 plays an important role. Indeed, this prompts our

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